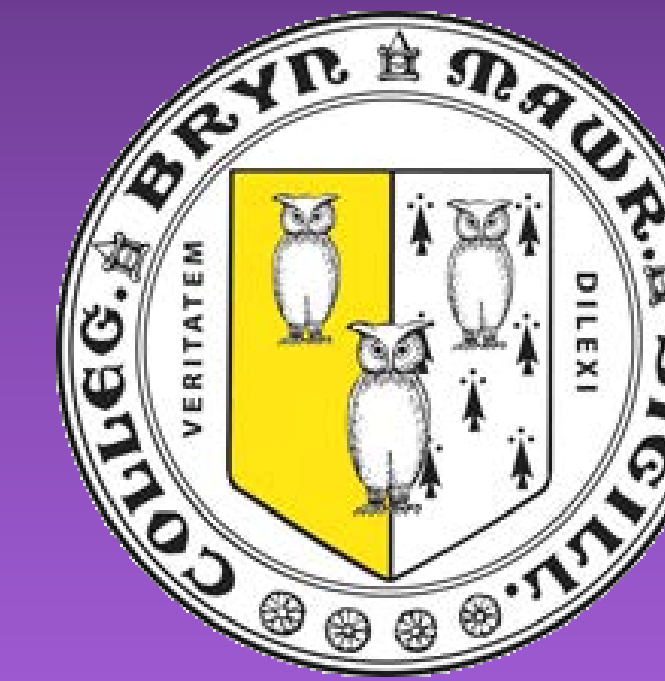


# Instantons in Quantum Mechanics

Genevieve Love, Avalon Vanis  
Mentored by Michael Schulz

Theoretical Physics, Bryn Mawr College



## Abstract

Our research looked at tunneling through a finite potential square barrier through the lens of instantons in Euclidean spacetime. We used the problem of a photon incident upon a sheet of glass as a model to connect Schrodinger's interpretation of tunneling to Sydney Coleman's instanton interpretation. In doing this, we looked at the Fourier transformation in order to change Coleman's fixed time propagator to a fixed energy propagator. Finally, we attempted to connect the Wentzel-Kramers-Brillouin (WKB) approximation for a simple harmonic oscillator<sup>2</sup> to Coleman's instanton approach for the standard double well potential.

## Introduction

The majority of our work was done with consideration to a finite square barrier with equation

$$V(x) = \begin{cases} 0, & x < -x_1 \\ V_0, & -x_1 < x < x_1 \\ 0, & x > x_1 \end{cases}$$

where  $V_0$  is an arbitrary potential and  $\pm x_1 \neq 0$ . However, because of the improbability of a perfect square barrier we approximated the barrier as smooth in our equations (fig. 1).

We wished to interpret tunneling through this barrier within the scope of instantons. An instanton is a mathematical expression of quantum tunneling that looks at the action in Euclidian time. Tunneling occurs when a particle passes through a barrier that it would not pass through in classical mechanics due to insufficient energy. The transmission amplitude gives the probability that the particle will pass through the barrier in quantum mechanics.

The Wentzel-Kramers-Brillouin (WKB) approximation was utilized to further examine Coleman's instanton interpretation of tunneling through the double potential well. This approximation is valid everywhere except the turning points. The area of the turning point is then approximated as a straight line and matched with the original equation using Airy functions.

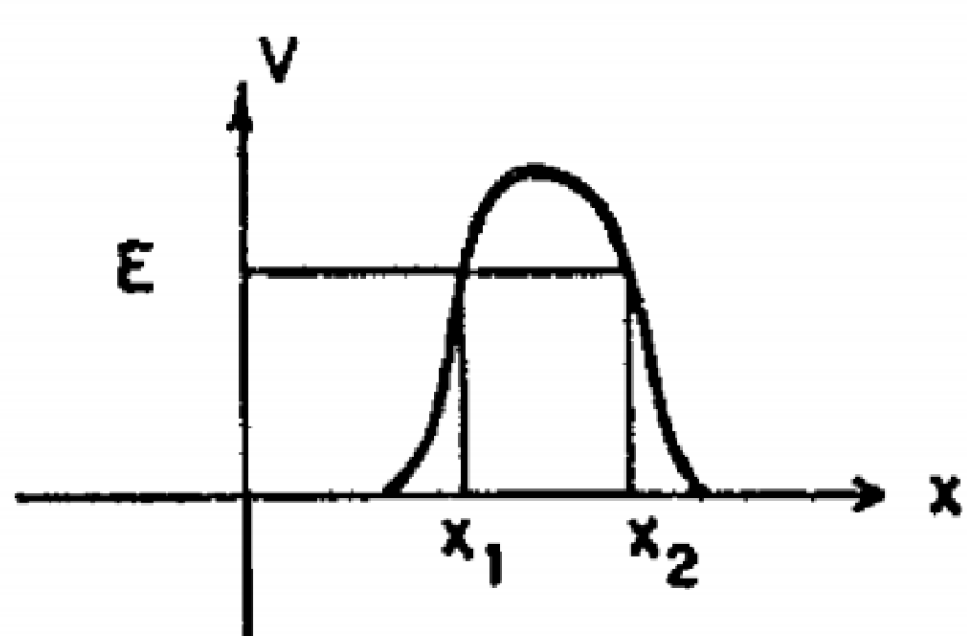


Fig 1. Exaggerated smooth approximation of the square potential barrier. " $x_1$ " and " $x_2$ " are the turning points.<sup>1</sup>

## Conclusions

Although we were unable to connect Coleman's work on instantons in the double well to our finding for the finite square barrier, we were able to gain a better understanding of the mathematical interpretations of tunneling. With further investigation into the methods Coleman used we believe we could recreate both the exact answer we found from the Schrodinger equation for the square potential and create a clear connection between the WKB result for the double harmonic oscillator and the result that Coleman found.

## Methods

We began by finding the probability of detection for a photon incident upon a glass plate. We then solved Schrodinger's equation for a square potential to find transmission amplitudes for energy greater than and less than the potential. Our goal then was to use the method outlined in Sidney Coleman's paper *The Uses of Instantons* to solve for the transmission amplitude in a square barrier.

Finally, we looked at the double well potential. We used the WKB formula to find a solution to the wave equation for a double harmonic oscillator with equation

$$(fig. 2). V(x) = \begin{cases} \frac{1}{2}m\omega^2(x+a)^2, & x < 0 \\ \frac{1}{2}m\omega^2(x-a)^2, & x > 0 \end{cases}$$

We then compared this result to the result Coleman found for the double well potential (fig. 3).

Fig. 2. (left) Example graph of the double harmonic oscillator used with the WKB formula.

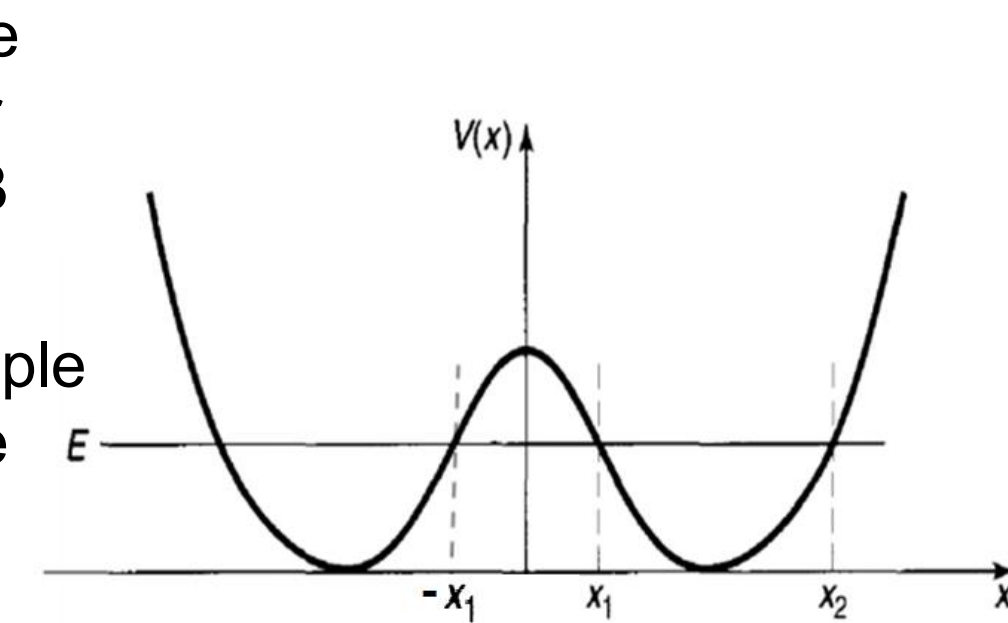
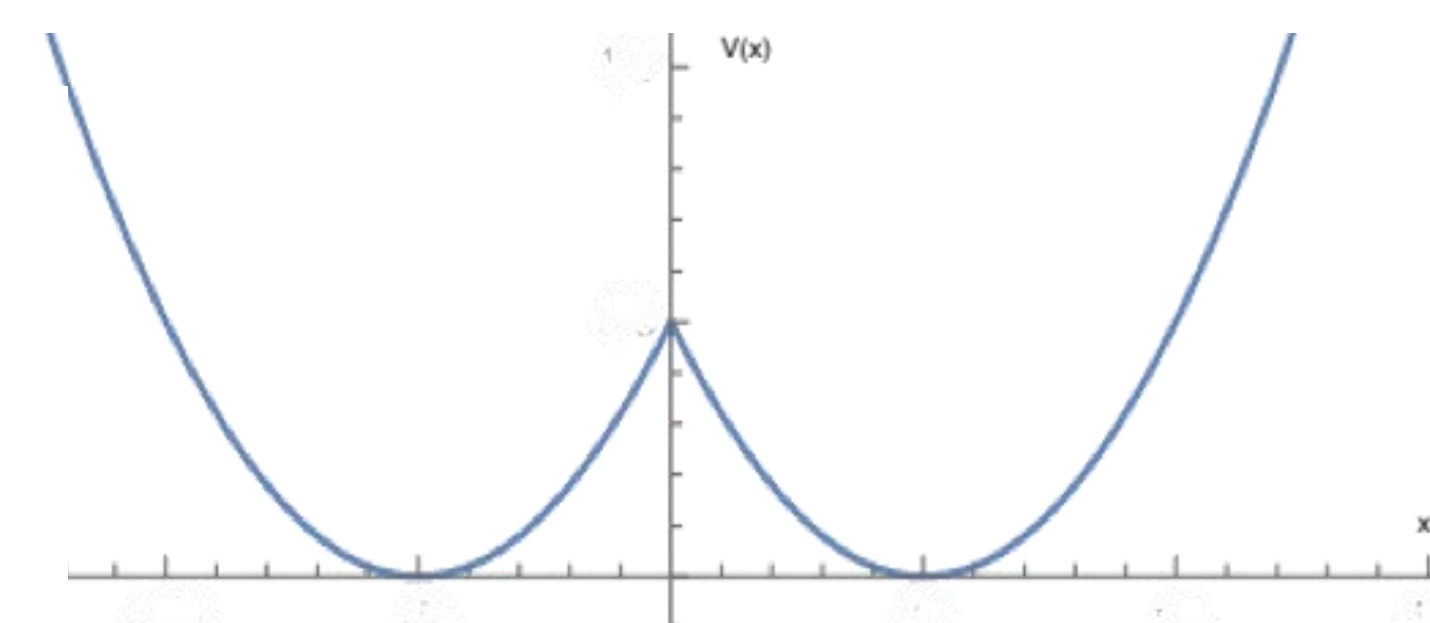


Fig. 3. (right) Example graph of the double potential well that Coleman used.<sup>2</sup>



## Discussion

In Coleman's paper he looked at a double potential well (fig. 3) whereas we were looking at a square potential barrier. This caused three major differences in our computations:

1. The double potential well allows the particle to sit at the minima of the potential for any amount of time whereas the particle incident upon the square barrier can never be at rest. This meant that, while Coleman used a fixed time to calculate his answer, we would have to look at fixed energy.
2. Because we were looking from directly before through directly after the tunneling occurred, we were working almost solely in Euclidian time.
3. Reflection and transmission amplitudes cannot be properly found because the barrier is vertical. We would need to find a way to calculate them while keeping the integrity of the square potential. This was done by approximating the square barrier as smooth (fig. 1). This was crucial because Coleman's work relied on the second derivative of the potential, which is not well-defined for a square barrier.

We believe the first point can be resolved through use of the Fourier transformation.

We believe the inability to create meaningful parallels between Coleman and the WKB formula was because

1. The WKB formula approximates the turning points (energy equal to potential) which can lead to inexact answers.
2. We were looking at a double harmonic oscillator which meant the derivative and second derivative were undefined at  $x = 0$ . The double potential well is smooth with well-defined derivatives at every point.

## Findings

In our example of a photon incident upon a glass plate, it went through three states. The photon was in its first state directly after being fired before it has encountered the glass plate. It then passed through the upper side of the plate and was inside the glass. When passing through the glass, the photon goes through a phase change and thus has a different wavelength in this second state. Once inside the glass, the photon may bounce between the upper and lower bounds of the glass any amount of times before escaping the lower bound and being in the third and final state below the glass. There is no phase change when the photon escapes the glass.

$$z_t = t e^{ikd_1} \tilde{t} e^{ikd_2} \sum_{m=0}^n r^{2m} e^{(2m+1)ik'd}$$

gives the probability amplitude  $z_t$  for the photon in which  $t$  is the transmission probability amplitude when incident upon the glass,  $\tilde{t}$  is the transmission amplitude probability when exiting the glass,  $k$  is the wave number of the photon outside the glass,  $k'$  is the wave number of the photon inside the glass,  $d_1$  is the distance the photon travels before it becomes incident upon the glass,  $d_2$  is the distance from exiting the glass to the detector,  $d$  is the thickness of the glass,  $r$  is the probability amplitude for reflection, and  $n$  is the number of times the photon reflects inside the glass.

In parallel to the optics problem, when a small particle is incident upon a potential barrier, such as the square barrier we were studying, it has a state before it reaches the barrier, it has a state after reaching the barrier where it experiences a phase change and can propagate back and forth any amount of times, and it has a state after exiting the barrier. Because of the symmetry between the two situations, it was easy to adapt the above example to fit a particle incident upon a square barrier when the particle had enough energy to pass. Thus, we used the time independent

Schrodinger Equation,  $\psi(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$ , to solve for the transmission amplitude of energy incident upon a square barrier in the allowed region (energy greater than the potential). This yielded

$$T = \left[ 1 + \frac{(k^2 - k_0^2)^2}{4k^2 k_0^2} \sin^2(k_0 a) \right]^{-1}$$

which is equal to the complex conjugate of  $z_t$ . Thus, the probability of detection for the photon and the transmission amplitude for the square barrier in the allowed region were equal. We now had an accurate classical mechanics comparison for energy incident upon a square barrier in this region.

We assumed that we would be able to use the optics example to better understand an instanton approach to tunneling through a square barrier in the forbidden (energy less than potential) region because the optics problem looks at the whole action instead of just one moment, and instantons are a way of looking at tunneling by looking at the whole action in Euclidian time. We used Coleman's paper as the main source for our efforts to construct the instanton interpretation for tunneling. However, we were unable to complete this portion of the research.

We were unable to create meaningful parallels between our WKB results and Coleman's interpretation of the double well.

## References

1. Coleman, S. (1979). *The Uses of Instantons*. The Whys of Subnuclear Physics, 805-941. doi:10.1007/978-1-4684-0991-8\_16
2. Griffiths, D. J. (1995). *Introduction to quantum mechanics* (First ed.). Upper Saddle River, NJ: Prentice-Hall.

## Acknowledgments

We would like to thank our research adviser Michael Schultz for all his guidance through the research process.