Compactification without Truncation of Yang Mills Action

Shiksha Pandey, Michael Schulz

Department of Physics, Bryn Mawr College

Abstract

We study the complete Kaluza-Klein expansion of Yang-Mills theory on a compact manifold.
The moduli space of Yang-Mills gauge potentials is a principal fiber bundle, whose metric determines the kinetic terms for charged scalar fields in the Kaluza-Klein expansion. We present an expression for the physical metric on moduli space in terms of fiber metric, compensator field (geometrically, the bundle connection) and base metric, taking motivations from the case of U(1) gauge theory. We then determine the corresponding Kaluza-Klein expansion of D-dimesional Yang-Mills action when compactified down to d-dimensions on the compact manifold Y. We avoid the use of privileged expansion bases to highlight geometric features rather than basis dependent simplifications.

Introduction

After Albert Einstein published his paper on the theory of relativity, Kaluza and Klein observed that 4D general relavity and 4D electromagnetism (plus a scalar field) can be obtained from the compactificatino of 5D gravity on a circle, when one truncates to lowest Fourier modes on the circle. This compactification followed by truncation is known as dimensional reduction. With this geometric unification, they pioneered a way to describe a d-dimensional theory in a simpler way using D>d-dimensions. In the dimensional reduction of gauge theories, it is necessary to introduce "compensator fields" (Lagrange multipliers introduced by hand) to preserve gauge invariance. In the case of K compactification, however, they appear on their own as connections on the fiber bundle.

Here, we will be identifying those compensators in case of U(1) gauge theory, and D-dimensional Yang-Mills theory--which is a generalization of the former to a non-abelian group—by studying their KK expansions.

U(1) Gauge Theory and Kaluza-Klein Expansion

The action of a U(1) gauge theory on a D-d dimensional compact manifold $\mathcal{Y}$ of metric $G^{(\mathcal{Y})}_{mn}$ is: $S = -\frac{1}{4e^2}\int d^{D-d}y\sqrt{G^{(\mathcal{Y})}}F^{mn}F_{mn}$, where $F_{mn} = \partial_m A_n - \partial_n A_m$.

Now, consider the full space $\mathcal{A}$ of gauge connections $A_m(y)$ on $\mathcal{Y}$. This space is a principal fiber bundle. The fiber $\mathcal{G}$ is the space of gauge transformations. The base $\mathcal{A}/\mathcal{G}$ is the physical space of equivalence classes of gauge connections and the physical metric on $\mathcal{A}$, $G_{\Lambda\Lambda'} = \int_{\mathcal{Y}} d^{D-d}y\sqrt{g_{D-d}}\, g^{mn}\,\partial_\Lambda A_m \partial_{\Lambda'} A_n$, can be expressed in the fiber bundle form as follows: $ds^2_{\mathcal{A}} = G_{IJ}(u)du^I du^J + G_{ll'}(dv^l + \Omega^l_I du^I)\left(dv^{l'} + \Omega_{l'J} du^J\right)$,

Where we have: $G_{ll'} = \int_{\mathcal{Y}} d^{D-d}y\sqrt{g_{D-d}}\, g^{mn}\,\partial_m Y_l \partial_n\ Y_{l'} = m_l^2\delta_{ll'}$,

$m_l^2\Omega_I = \int_{\mathcal{Y}} d^{D-d}y\sqrt{g_{D-d}}\, g^{mn}\,\partial_I A_m \partial_n Y_l$,

$G_{IJ} = \int_{\mathcal{Y}} d^{D-d}y\sqrt{g_{D-d}}\, g^{mn}\,\delta_I A_m \delta_J A_n$, where $\nabla^2_{\mathcal{Y}} Y_l = -m_l^2 Y_l$ and $\delta_I A_m(u;y) = \partial_I A_m(u;y) - \Omega_{Im}(u;y)$ is the horizontal derivative on the bundle. Here, we didn't have to introduce the compensator by hand; it was present in the physical metric already as the bundle connection.

In the full untruncated theory, $S = \int d^D x\sqrt{-g_D}(-\frac{1}{4}F^{mn}F_{mn})$ $= \int d^D x\sqrt{-g_D}(\mathcal{L}_{gauge} + \mathcal{L}_{charged\ scaler} + \mathcal{L}_{neutral\ scaler})$, the bosons eat the vertical moduli, and the scalers deforming away from flat F_{mn}=0 are massive.

KK Expansion in Yang-Mills Theory

Notation and Geometry

We will follow similar steps as in U(1) case. So we proceed keeping our considerations same as before.

Given a group G, the lie algebra $\mathfrak{g}$ is given by: $[t_a, t_b] = if^c_{ab}t_c$. Given a basis $\{t_a\}$ of $\mathfrak{g}$, and v^a as the coordinates on G, $g(v) = e^{(iv^a t_a)}$. Given a basis $\{f_a(y)\}$ of $\mathfrak{g}$-valued functions on $\mathcal{Y}$, $g(v;y) = e^{(iv^\alpha f_\alpha(y))}$.

For group $\mathcal{G}$ of gauge transformations, the lie algebra is then: $[T_\alpha, T_\beta] = if^\gamma_{\alpha\beta}T_\gamma$.

So $[f_\alpha(y), f_\beta(y)] = if^\gamma_{\alpha\beta}f_\gamma(y)$.

Letting u^I denote the coordinates on $\mathcal{A}/\mathcal{G}$, $A_m = g^{-1}(v;y)A^0_m(u;y)g(v;y) + \theta(v;y)\partial_m v^\alpha(y)$, where $A^0_m(u;y)$ denotes a fiducial representative of each gauge field, and $\theta_\alpha(v;y)$= $-ig^{-1}(v;y)d_{\mathcal{G}}\, g(v;y)$. Now, let $\Theta(v)$ be the left invariant 1-form on $\mathcal{G}$. Then, $\theta(v;y) = \Theta^\alpha(v)f_\alpha(y)$.

The physical metric resulted from the compactification of the Yang-Mills action is then: $ds^2 = \int_{\mathcal{Y}} d^{D-d}y\sqrt{G^{(\mathcal{Y})}}tr[d_{\mathcal{G}}A_m(u,v;y)\, d_{\mathcal{G}}A_n(u,v;y)$.

KK Action without Truncation

Letting $\widetilde{\Theta}(v) = \widetilde{\Theta^\alpha_\beta}(v)T_\alpha dv^\beta$ denote the canonical R-invariant form on $\mathcal{G}$,

$ds^2 = G_{IJ}(u)du^I du^J + G_{\alpha\beta}\left(\widetilde{\Theta^\alpha_\gamma}(v)dv^\gamma + \Omega^\alpha_I(u)du^I\right)\left(\widetilde{\Theta^\beta_\delta}(v)dv^\delta + \Omega^\beta_J(u)du^J\right)$.

where, $G_{\alpha\beta}(u) = \int_{\mathcal{Y}} d^{D-d}y\sqrt{G^{(\mathcal{Y})}}tr\left[f_\alpha(y)\left(-(D^0)^2\right)f_\beta(y)\right]$,

$\Omega^\alpha_I(u) = G^{\alpha\beta}(u)\mathcal{J}_{\beta I}(u)$,

$\mathcal{J}_{\beta I}(u) = \int_{\mathcal{Y}} d^{D-d}y\sqrt{G^{(\mathcal{Y})}}G^{(y)mn}tr[f_\beta(y)\,\partial_I A^0_m\,(u;y)]$,

$G_{IJ}(u) = \int_{\mathcal{Y}} d^{D-d}y\sqrt{G^{(\mathcal{Y})}}G^{(y)mn}tr[\partial^{hor}_I A_m \partial^{hor}_J A_n]$, with $\partial^{hor}_I A_m(u,v;y) = \partial_I A_m(u,v;y) - D_m\Omega_I(u;y)$, and $D_m\phi(y) = \partial_m\phi(y) - i[A_m, \phi(x)]$.

We then promote u, v to u(x), v(x) in compactification, and obtain the following components of the field strength: $F_{\mu\nu}(x,y) = F^\alpha_{\mu\nu}(x)f_\alpha(y)$,

$F_{\mu m}(x,y) = \partial^{hor}_I A_m(u,v;y) + D_m(\widetilde{D_\mu}v^\alpha(x)f_a(y))$,

$F_{mn}(x,y) = F_{mn}(u,v;y) = g^{-1}(v;y)F^0_{mn}(u;y)g(v,y)$,

where $\widetilde{D_\mu}v^\alpha(x) = \widetilde{\Theta^\alpha_\gamma}(v(x))\partial_\mu v^\gamma(x) - A^\alpha_\mu(x)$.

Substituting these components into the D-dimensional Yang-Mills action:, we get the same result as in U(1) case: $S = \int d^D x\sqrt{-G^{(d)}}(\mathcal{L}_{gauge} + \mathcal{L}_{charged\ scaler} + \mathcal{L}_{neutral\ scaler})$,

where $\mathcal{L}_{gauge} = -\frac{1}{4g^2}d_{\alpha\beta}F^{\alpha\mu\nu}F^\beta_{\mu\nu}$,

$\mathcal{L}_{charged\ scaler} = -\frac{1}{2g^2}G_{\alpha\beta}D^\mu v^\alpha(x)D_\mu v^\beta(x)$,

$\mathcal{L}_{neutral\ scaler} = -\frac{1}{2g^2}G_{IJ}(u)\partial^\mu u^I(x)\partial_\mu u^J(x) - V(u(x))$,

and $d_{\alpha\beta} = \int_{\mathcal{Y}} d^{D-d}y\sqrt{G^{(\mathcal{Y})}}tr[f_\alpha(y)f_\beta(y)]$,

$G_{\alpha\beta}(u) = \int_{\mathcal{Y}} d^{D-d}y\sqrt{G^{(\mathcal{Y})}}tr\left[f_\alpha(y)\left(-(D^0)^2\right)f_\beta(y)\right]$,

$D^\mu v^\alpha = \left(\widetilde{\Theta^\alpha_\gamma}(v)\partial_\mu v^\gamma + \Omega^\alpha_I(u)\partial_\mu u^I - A^\alpha_\mu(x)\right)$,

$V(u) = -\frac{1}{4g^2}d_{\alpha\beta}\int_{\mathcal{Y}} d^{D-d}\sqrt{G^{(\mathcal{Y})}}tr[F^{mn}(u;y)F_{mn}(u;y)]$.

As in the U(1) case, the gauge field $c_\alpha A^\alpha_\mu(x)$ eats $c_\alpha\left(\widetilde{\Theta^\alpha_\gamma}(v)\partial_\mu v^\gamma + \Omega^\alpha_I(u)\partial_\mu u^I\right)$ to become massive vector of mass m, and the scalers u^I deforming away from $F_{mn}(u;y)$=0 are massive due to V(u).

In contract to the U(1) case, the covariant laplace operator depends on u, so its eigenfunctions no longer provide convenient expansion basis, except on a single fiber of a moduli space.

Discussion and Conclusion

- As seen in both the U(1), and Yang-Mills theory, consistent dimensional reduction requires introduction of compensators explicitly, whereas for the untruncated theory, that is not necessary as the compensators are already present in the moduli space metric.
- For Yang-Mills theory, we determined the physical metric on moduli space and found that its vertical metric $G_{\alpha\beta}$ is an R-invariant metric determined by the gauge covariant Laplacian on $\mathcal{Y}$, not the standard bi-invariant metric.
- In the full untruncated theory, the KK gauge bosons eat the vertical moduli, and the scalar potential lifts those horizontal moduli corresponding to non-flat deformations of the gauge field.
- The effective field theory below the compactification scale exactly agrees with that of the dimensional reduction ansatz. The U(1) and Yang-Mills stories are analogous, but a noteworthy difference is that there is no correspondingly simple global expansion in Laplace eigenfunctions in the Yang-Mills case

References

1. D. Bailin, A. Love, "Kaluza-Klein theories", Rep. Prog. Phys. 50 (1987)
2. M. R. Douglas, G. Torroba, "Kinetic terms in warped compactifications", JHEP 0905:013 (2009)
3. M. Nakahara, "Geometry, Topology, and Physics", Ch. 10, ISBN 0 7503 0606 8 (2003)

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