Optimal Control Theory

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Abstract

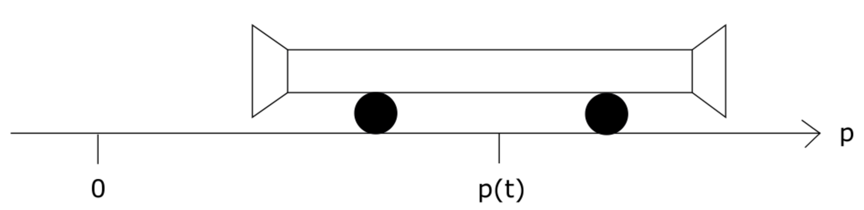
Optimal Control theory is an extension of Calculus of Variations that deals with finding a control law so that a certain optimality criterion is achieved. For example, consider a car driving straight on a hilly road. A natural question is: how should the driver press the accelerator pedal in order to minimize the total travel time given constraints such as the amount of fuel and speed limits? The control law would tell you the optimal way to press the accelerator and shift the gears.

This summer, we focused on optimal control problems for ordinary differential equations. We studied the necessary conditions for this kind of problem: Pontryagin's Principle. We then applied it to real world problems.

Introduction

Optimal Control theory is an extension of Calculus of Variations that deals with finding a control law so that a certain optimality criterion is achieved.

Here is a sample problem. Suppose you have a rocket car with two rocket engines, one at the left end and the other at the right end, as illustrated below:



The goal is to move the car to a specified target position. Let *p(t)* denote the **position** the car is in at time *t*, and let *p'(t)* denote the **velocity** of the car at time *t*. Then the **state** of the car is the vector *x(t) = (p(t),p'(t))*. Suppose our target state is *(0,0)*. The **control** of this state, denoted as *u(t)*, is the **force** on the car due to firing either engine at time *t*. If we fire the right engine, *u(t)* is negative; if we fire the left engine, *u(t)* is positive. A natural problem is to try to move the car to the desired target position while minimizing or maximizing some other quantity, such as minimizing the amount of fuel used over the entire time. Quantities to be maximized or minimized can be expressed as an integral.

In general, the basic optimal control problem is to find a piecewise continuous control *u(t)* and the associated state variable *x(t)* to optimize a given functional:

(**) $$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))dt$$

$$\text{subject to } x'(t) = g(t, x(t), u(t)),\ x(t_0) = x_0, \text{and } x(t_1) \text{ free}$$

The control *u(t)* satisfying this objective functional is called the **optimal control**.

Methods

To solve an optimal control problem, we employ the following theorem, often called **Pontryagin's Principle**. This will give us some necessary conditions that need to be satisfied by the optimal control, *u*(t)*, and the optimal state, *x*(t)*. We can apply this theorem to get *candidates* for the optimal control; additional theorems show the existence of optimal controls.

Theorem 4.1. *If $u^*(t)$ and $x^*(t)$ are optimal for the basic optimal control problem ((**)), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leqslant H(t, x^*(t), u^*(t), \lambda(t)) \tag{4.1}$$

for all controls u at each time t, where the Hamiltonian H is

$$H = f(t, x(t), u(t)) + \lambda(t) g(t, x(T), u(t)), \tag{4.2}$$

$$\text{and} \tag{4.3}$$

$$\lambda'(t) = -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}, \tag{4.4}$$

$$\lambda(t_1) = 0 \tag{4.5}$$

Here is the outline to use Pontryagin Principle to solve an optimal problem:

1. Form the Hamiltonian for the problem
2. Write the adjoint differential equation, transversality boundary condition, and the optimality condition.
3. Try to eliminate *u** by using optimality equation $H_u = 0$. Solve for *u** in terms of *x** and *λ*.
4. Solve two differential equations for *x** and *λ* with two boundary conditions. Substitute *u** in the differential equation with the expression for the optimal control from the previous step.
5. After finding the optimal state and adjoint, solve for optimal control.

An illustration of this outline is followed in the Results section. In simple examples, one can solve for the optimal control by hand. In general, one can write MATLAB programs to solve these equations.

Discussion

Optimal control problems arise in many systems, including Ordinary differential equations, partial differential equations, discrete equations, stochastic differential equations, integro-difference equations, and combination of discrete and continuous systems. We have focused on optimal control problem for ordinary differential equations.

Also, in real life phenomena, differential control systems are often established with delays. Delays in time will postpone the input signal, and this might cause the control system lose stability. My future research will be on time-delay control systems.

Results

Application

Example 4.3. Solve

$$\min_u \int_1^2 tu(t)^2 + t^2 x(t) dt \tag{4.17}$$

$$\text{with } x'(t) = -u(t), x(1) = 1, x(2) \text{ free} \tag{4.18}$$

To solve this problem,let's follow the outline.

First, form the Hamiltonian:

$$H = f + \lambda g \tag{4.19}$$

$$= tu(t)^2 + t^2 x(t) + (-\lambda u(t)) \tag{4.20}$$

The optimality condition is:

$$0 = \frac{\partial H}{\partial u} \tag{4.21}$$

$$= 2tu(t) - \lambda \text{ at } u^* \tag{4.22}$$

solve for u^* through optimality condition, we have:

$$u^* = \frac{\lambda}{2t} \tag{4.23}$$

Now we need to check if this is a maximizing or a minimizing problem. We have

$$\frac{\partial^2 H}{\partial u^2} = 2t > 0 \text{ since } t \in [0, \infty) \tag{4.24}$$

Thus this is a minimizing problem.

The detailed solution is omitted here, but we can have a sense of how Pontryagin's principle can be used to solve a control problem. It is also widely used in real-life system like Mold-Fungicide Model and Bacteria Model. We use Matlab to simulate the process of solving those models.

Conclusions

Optimal Control Problems are frequently seen in industries. We prove Pontryagin Principle and derive optimal control problems' necessary conditions through it. By computer programming, we can simulate the process of solving complicated optimal control problems.

References

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